

Functoriality and K -theory for $\mathrm{GL}_n(\mathbb{R})$

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Abstract

We investigate base change and automorphic induction \mathbb{C}/\mathbb{R} at the level of K -theory for the general linear group $\mathrm{GL}_n(\mathbb{R})$. In the course of this study, we compute in detail the C^* -algebra K -theory of this disconnected group. We investigate the interaction of base change with the Baum-Connes correspondence for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. This article is the archimedean companion of our previous article [11].

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1 Introduction

In the general theory of automorphic forms, an important role is played by *base change* and *automorphic induction*, two examples of the principle of functoriality in the Langlands program [3]. Base change and automorphic induction have a global aspect and a local aspect [1][7]. In this article, we focus on the archimedean case of base change and automorphic induction for the general linear group $\mathrm{GL}(n, \mathbb{R})$, and we investigate these aspects of functoriality at the level of K -theory.

For $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ we have the Langlands classification and the associated L -parameters [8]. We recall that the domain of an L -parameter of $\mathrm{GL}_n(F)$ over an archimedean field F is the Weil group W_F . The Weil groups are given by

$$W_{\mathbb{C}} = \mathbb{C}^{\times}$$

and

$$W_{\mathbb{R}} = \langle j \rangle \mathbb{C}^{\times}$$

where $j^2 = -1 \in \mathbb{C}^{\times}$, $jc = \bar{c}j$ for all $c \in \mathbb{C}^{\times}$. Base change is defined by restriction of L -parameter from $W_{\mathbb{R}}$ to $W_{\mathbb{C}}$.

An L -parameter ϕ is *tempered* if $\phi(W_F)$ is bounded. Base change therefore determines a map of tempered duals.

In this article, we investigate the interaction of base change with the Baum-Connes correspondence for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$.

Let F denote \mathbb{R} or \mathbb{C} and let $G = G(F) = \mathrm{GL}_n(F)$. Let $C_r^*(G)$ denote the reduced C^* -algebra of G . The Baum-Connes correspondence is a canonical isomorphism [2][5][9]

$$\mu_F : K_*^{G(F)}(\underline{EG}(F)) \rightarrow K_* C_r^*(G(F))$$

where $\underline{EG}(F)$ is a universal example for the action of $G(F)$.

The noncommutative space $C_r^*(G(F))$ is strongly Morita equivalent to the commutative C^* -algebra $C_0(\mathcal{A}_n^t(F))$ where $\mathcal{A}_n^t(F)$ denotes the tempered dual of $G(F)$, see [12, §1.2][13]. As a consequence of this, we have

$$K_* C_r^*(G(F)) \cong K^* \mathcal{A}_n^t(F).$$

This leads to the following formulation of the Baum-Connes correspondence:

$$K_*^{G(F)}(\underline{EG}(F)) \cong K^* \mathcal{A}_n^t(F).$$

Base change and automorphic induction \mathbb{C}/\mathbb{R} determine maps

$$\mathcal{BC}_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{R}) \rightarrow \mathcal{A}_n^t(\mathbb{C})$$

and

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{C}) \rightarrow \mathcal{A}_{2n}^t(\mathbb{R}).$$

This leads to the following diagrams

$$\begin{array}{ccc} K_*^{G(\mathbb{C})}(\underline{EG}(\mathbb{C})) & \xrightarrow{\mu_{\mathbb{C}}} & K^* \mathcal{A}_n^t(\mathbb{C}) \\ \downarrow & & \downarrow \mathcal{BC}_{\mathbb{C}/\mathbb{R}}^* \\ K_*^{G(\mathbb{R})}(\underline{EG}(\mathbb{R})) & \xrightarrow{\mu_{\mathbb{R}}} & K^* \mathcal{A}_n^t(\mathbb{R}). \end{array}$$

and

$$\begin{array}{ccc} K_*^{G(\mathbb{R})}(\underline{EG}(\mathbb{R})) & \xrightarrow{\mu_{\mathbb{R}}} & K^* \mathcal{A}_{2n}^t(\mathbb{R}) \\ \downarrow & & \downarrow \mathcal{AI}_{\mathbb{C}/\mathbb{R}}^* \\ K_*^{G(\mathbb{C})}(\underline{EG}(\mathbb{C})) & \xrightarrow{\mu_{\mathbb{C}}} & K^* \mathcal{A}_n^t(\mathbb{C}). \end{array}$$

where the left-hand vertical maps are the unique maps which make the diagrams commutative.

In section 2 we describe the tempered dual $\mathcal{A}_n^t(F)$ as a locally compact Hausdorff space.

In section 3 we compute the K -theory for the reduced C^* -algebra of $\mathrm{GL}(n, \mathbb{R})$. The real reductive Lie group $\mathrm{GL}(n, \mathbb{R})$ is not connected. If n is even our formulas show that we always have non-trivial K^0 and K^1 . We also recall the K -theory for the reduced C^* -algebra of the complex reductive group $\mathrm{GL}(n, \mathbb{C})$, see [13]. In section 4 we recall the Langlands parameters for $\mathrm{GL}(n)$ over archimedean local fields, see [8]. In section 5 we compute the base change map $\mathcal{BC} : \mathcal{A}_n^t(\mathbb{R}) \rightarrow \mathcal{A}_n^t(\mathbb{C})$ and prove that \mathcal{BC} is a continuous proper map. At the level of K -theory, base change is the zero map for $n > 1$ (Theorem 5.4) and is nontrivial for $n = 1$ (Theorem 5.5). In section 6, we compute the automorphic induction map $\mathcal{AI} : \mathcal{A}_n^t(\mathbb{C}) \rightarrow \mathcal{A}_{2n}^t(\mathbb{R})$. Contrary to base change, at the level of K -theory, automorphic induction is nontrivial for every n (Theorem 6.5). In section 7, where we study the case $n = 1$, base change for K^1 creates a map

$$\mathcal{R}(\mathrm{U}(1)) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{R}(\mathrm{U}(1))$ is the representation ring of the circle group $\mathrm{U}(1)$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $\mathrm{U}(1)$ to $1 \oplus \varepsilon$, where ε is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $\mathrm{U}(1)$ to zero.

This map has an interpretation in terms of K -cycles. The K -cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to \mathbb{C}^\times and \mathbb{R}^\times , and therefore determines a class $\mathcal{J}_{\mathbb{C}} \in K_1^{\mathbb{C}^\times}(\underline{E}\mathbb{C}^\times)$ and a class $\mathcal{J}_{\mathbb{R}} \in K_1^{\mathbb{R}^\times}(\underline{E}\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\mathcal{J}_{\mathbb{C}} \mapsto (\mathcal{J}_{\mathbb{R}}, \mathcal{J}_{\mathbb{R}})$$

This extends the results of [11] to archimedean fields.

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2 On the tempered dual of $\mathrm{GL}(n)$

Let $F = \mathbb{R}$. In order to compute the K -theory of the reduced C^* -algebra of $\mathrm{GL}(n, F)$ we need to parametrize the tempered dual $\mathcal{A}_n^t(F)$ of $\mathrm{GL}(n, F)$.

Let M be a standard Levi subgroup of $\mathrm{GL}(n, F)$, i.e. a block-diagonal subgroup. Let 0M be the subgroup of M such that the determinant of each block-diagonal is ± 1 . Denote by $X(M) = \widehat{M/{}^0M}$ the group of *unramified characters* of M , consisting of those characters which are trivial on 0M .

Let $W(M) = N(M)/M$ denote the Weyl group of M . $W(M)$ acts on the discrete series $E_2({}^0M)$ of 0M by permutations.

Now, choose one element $\sigma \in E_2({}^0M)$ for each $W(M)$ -orbit. The *isotropy subgroup* of $W(M)$ is defined to be

$$W_\sigma(M) = \{\omega \in W(M) : \omega.\sigma = \sigma\}.$$

Form the disjoint union

$$\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = \bigsqcup_M \bigsqcup_{\sigma \in E_2({}^0M)} X(M)/W_\sigma(M). \quad (1)$$

The disjoint union has the structure of a locally compact, Hausdorff space and is called the *Harish-Chandra parameter space*. The parametrization of the tempered dual $\mathcal{A}_n^t(\mathbb{R})$ is due to Harish-Chandra, see [10].

Proposition 2.1. *There exists a bijection*

$$\begin{array}{ccc} \bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) & \longrightarrow & \mathcal{A}_n^t(\mathbb{R}) \\ \chi^\sigma & \mapsto & i_{GL(n),MN}(\chi^\sigma \otimes 1), \end{array}$$

where $\chi^\sigma(x) := \chi(x)\sigma(x)$ for all $x \in M$.

In view of the above bijection, we will denote the Harish-Chandra parameter space by $\mathcal{A}_n^t(\mathbb{R})$.

We will see now the particular features of the archimedean case, starting with $GL(n, \mathbb{R})$. Since the discrete series of $GL(n, \mathbb{R})$ is empty for $n \geq 3$, we only need to consider partitions of n into 1's and 2's. This allows us to decompose n as $n = 2q + r$, where q is the number of 2's and r is the number of 1's in the partition. To this decomposition we associate the partition

$$n = (\underbrace{2, \dots, 2}_q, \underbrace{1, \dots, 1}_r),$$

which corresponds to the Levi subgroup

$$M \cong \underbrace{GL(2, \mathbb{R}) \times \dots \times GL(2, \mathbb{R})}_q \times \underbrace{GL(1, \mathbb{R}) \times \dots \times GL(1, \mathbb{R})}_r.$$

Varying q and r we determine a representative in each equivalence class of Levi subgroups. The subgroup 0M of M is given by

$${}^0M \cong \underbrace{SL^\pm(2, \mathbb{R}) \times \dots \times SL^\pm(2, \mathbb{R})}_q \times \underbrace{SL^\pm(1, \mathbb{R}) \times \dots \times SL^\pm(1, \mathbb{R})}_r,$$

where

$$SL^\pm(m, \mathbb{R}) = \{g \in GL(m, \mathbb{R}) : |\det(g)| = 1\}$$

is the *unimodular subgroup* of $GL(m, \mathbb{R})$. In particular, $SL^\pm(1, \mathbb{R}) = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

The representations in the discrete series of $GL(2, \mathbb{R})$, denoted \mathcal{D}_ℓ for $\ell \in \mathbb{N}$ ($\ell \geq 1$) are induced from $SL(2, \mathbb{R})$ [8, p.399]:

$$\mathcal{D}_\ell = \text{ind}_{SL^\pm(2, \mathbb{R}), SL(2, \mathbb{R})}(\mathcal{D}_\ell^\pm),$$

where \mathcal{D}_ℓ^\pm acts in the space

$$\{f : \mathcal{H} \rightarrow \mathbb{C} | f \text{ analytic}, \|f\|^2 = \int \int |f(z)|^2 y^{\ell-1} dx dy < \infty\}.$$

Here, \mathcal{H} denotes the Poincaré upper half plane. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\mathcal{D}_\ell^\pm(g)(f(z)) = (bz + d)^{-(\ell+1)} f\left(\frac{az + c}{bz + d}\right).$$

More generally, an element σ from the discrete series $E_2(^0M)$ is given by

$$\sigma = i_{G, MN}(\mathcal{D}_{\ell_1}^\pm \otimes \dots \otimes \mathcal{D}_{\ell_q}^\pm \otimes \tau_1 \otimes \dots \otimes \tau_r \otimes 1), \quad (2)$$

where $\mathcal{D}_{\ell_i}^\pm$ ($\ell_i \geq 1$) are the discrete series representations of $SL^\pm(2, \mathbb{R})$ and τ_j is a representation of $SL^\pm(1, \mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, i.e. $id = (x \mapsto x)$ or $sgn = (x \mapsto \frac{x}{|x|})$.

Finally we will compute the unramified characters $X(M)$, where M is the Levi subgroup associated to the partition $n = 2q + r$.

Let $x \in GL(2, \mathbb{R})$. Any character of $GL(2, \mathbb{R})$ is given by

$$\chi(\det(x)) = (sgn(\det(x)))^\varepsilon |\det(x)|^{it}$$

($\varepsilon = 0, 1, t \in \mathbb{R}$) and it is unramified provided that

$$\chi(\det(g)) = \chi(\pm 1) = (\pm 1)^\varepsilon = 1, \text{ for all } g \in SL^\pm(2, \mathbb{R}).$$

This implies $\varepsilon = 0$ and any unramified character of $GL(2, \mathbb{R})$ has the form

$$\chi(x) = |\det(x)|^{it}, \text{ for some } t \in \mathbb{R}. \quad (3)$$

Similarly, any unramified character of $GL(1, \mathbb{R}) = \mathbb{R}^\times$ has the form

$$\xi(x) = |x|^{it}, \text{ for some } t \in \mathbb{R}. \quad (4)$$

Given a block diagonal matrix $\text{diag}(g_1, \dots, g_q, \omega_1, \dots, \omega_r) \in M$, where $g_i \in GL(2, \mathbb{R})$ and $\omega_j \in GL(1, \mathbb{R})$, we conclude from (3) and (4) that any unramified character $\chi \in X(M)$ is given by

$$\begin{aligned} \chi(\text{diag}(g_1, \dots, g_q, \omega_1, \dots, \omega_r)) &= \\ &= |\det(g_1)|^{it_1} \times \dots \times |\det(g_q)|^{it_q} \times |\omega_1|^{it_{q+1}} \times \dots \times |\omega_r|^{it_{q+r}}, \end{aligned}$$

for some $(t_1, \dots, t_{q+r}) \in \mathbb{R}^{q+r}$. We can denote such element $\chi \in X(M)$ by $\chi_{(t_1, \dots, t_{q+r})}$. We have the following result.

Proposition 2.2. *Let M be a Levi subgroup of $GL(n, \mathbb{R})$, associated to the partition $n = 2q + r$. Then, there is a bijection*

$$X(M) \rightarrow \mathbb{R}^{q+r}, \quad \chi_{(t_1, \dots, t_{q+r})} \mapsto (t_1, \dots, t_{q+r}).$$

Let us consider now $GL(n, \mathbb{C})$. The tempered dual of $GL(n, \mathbb{C})$ comprises the *unitary principal series* in accordance with Harish-Chandra [6, p. 277]. The corresponding Levi subgroup is a maximal torus $T \cong (\mathbb{C}^\times)^n$. It follows that ${}^0T \cong \mathbb{T}^n$ the compact n -torus.

The principal series representations are given by

$$\pi_{\ell, it} = i_{G, TU}(\sigma \otimes 1), \tag{5}$$

where $\sigma = \sigma_1 \otimes \dots \otimes \sigma_n$ and $\sigma_j(z) = (\frac{z}{|z|})^{\ell_j} |z|^{it_j}$ ($\ell_j \in \mathbb{Z}$ and $t_j \in \mathbb{R}$).

An unramified character is given by

$$\chi \left(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \right) = |z_1|^{it_1} \times \dots \times |z_n|^{it_n}$$

and we can represent χ as $\chi_{(t_1, \dots, t_n)}$. Therefore, we have the following result.

Proposition 2.3. *Denote by T the standard maximal torus in $GL(n, \mathbb{C})$. There is a bijection*

$$X(T) \rightarrow \mathbb{R}^n, \quad \chi_{(t_1, \dots, t_n)} \mapsto (t_1, \dots, t_n).$$

3 K -theory for $GL(n)$

Using the Harish-Chandra parametrization of the tempered dual of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ (recall that the Harish-Chandra parameter space is a locally compact, Hausdorff topological space) we can compute the K -theory of the reduced C^* -algebras $C_r^*GL(n, \mathbb{R})$ and $C_r^*GL(n, \mathbb{C})$.

3.1 K -theory for $\mathrm{GL}(n, \mathbb{R})$

We exploit the strong Morita equivalence described in [12, §1.2]. We infer that

$$\begin{aligned} K_j(C_r^* \mathrm{GL}(n, \mathbb{R})) &= K^j(\bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M)) \\ &= \bigoplus_{(M, \sigma)} K^j(X(M)/W_\sigma(M)) \\ &= \bigoplus_{(M, \sigma)} K^j(\mathbb{R}^{n_M}/W_\sigma(M)), \end{aligned} \quad (6)$$

where $n_M = q + r$ if M is a representative of the equivalence class of Levi subgroup associated to the partition $n = 2q + r$. Hence the K -theory depends on n and on each Levi subgroup.

To compute (6) we have to consider the following orbit spaces:

- (i) \mathbb{R}^n , in which case $W_\sigma(M)$ is the trivial subgroup of the Weil group $W(M)$;
- (ii) \mathbb{R}^n/S_n , where $W_\sigma(M) = W(M)$ (this is one of the possibilities for the partition of n into 1's);
- (iii) $\mathbb{R}^n/(S_{n_1} \times \dots \times S_{n_k})$, where $W_\sigma(M) = S_{n_1} \times \dots \times S_{n_k} \subset W(M)$ (see the examples below).

Definition 3.1. *An orbit space as indicated in (ii) and (iii) is called a closed cone.*

The K -theory for \mathbb{R}^n may be summarized as follows

$$K^j(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \text{if } n = j \pmod{2} \\ 0 & \text{otherwise} \end{cases}.$$

The next results show that the K -theory of a closed cone vanishes.

Lemma 3.2. $K^j(\mathbb{R}^n/S_n) = 0, j = 0, 1.$

Proof. We need the following definition. A point $(a_1, \dots, a_n) \in \mathbb{R}^n$ is called normalized if $a_j \leq a_{j+1}$, for $j = 1, 2, \dots, n-1$. Therefore, in each orbit there is exactly one normalized point and \mathbb{R}^n/S_n is homeomorphic to the subset of \mathbb{R}^n consisting of all normalized points of \mathbb{R}^n . We denote the set of all normalized points of \mathbb{R}^n by $N(\mathbb{R}^n)$.

In the case of $n = 2$, let (a_1, a_2) be a normalized point of \mathbb{R}^2 . Then, there is a unique $t \in [1, +\infty[$ such that $a_2 = ta_1$ and the map

$$\mathbb{R} \times [1, +\infty[\rightarrow N(\mathbb{R}^2), (a, t) \mapsto (a, ta)$$

is a homeomorphism.

If $n > 2$ then the map

$$N(\mathbb{R}^{n-1}) \times [1, +\infty[\rightarrow N(\mathbb{R}^n), (a_1, \dots, a_{n-1}, t) \mapsto (a_1, \dots, a_{n-1}, ta_n)$$

is a homeomorphism. Since $[1, +\infty[$ kills both the K -theory groups K^0 and K^1 , the result follows by applying Künneth formula. \square

The symmetric group S_n acts on \mathbb{R}^n by permuting the components. This induces an action of any subgroup $S_{n_1} \times \dots \times S_{n_k}$ of S_n on \mathbb{R}^n . Write

$$\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \times \mathbb{R}^{n_k} \times \mathbb{R}^{n-n_1-\dots-n_k}.$$

If $n = n_1 + \dots + n_k$ then we simply have $\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k}$.

The group $S_{n_1} \times \dots \times S_{n_k}$ acts on \mathbb{R}^n as follows.

S_{n_1} permutes the components of \mathbb{R}^{n_1} leaving the remaining fixed;

S_{n_2} permutes the components of \mathbb{R}^{n_2} leaving the remaining fixed;

and so on. If $n > n_1 + \dots + n_k$ the components of $\mathbb{R}^{n-n_1-\dots-n_k}$ remain fixed.

This can be interpreted, of course, as the action of the trivial subgroup. As a consequence, one identifies the orbit spaces

$$\mathbb{R}^n / (S_{n_1} \times \dots \times S_{n_k}) \cong \mathbb{R}^{n_1} / S_{n_1} \times \dots \times \mathbb{R}^{n_k} / S_{n_k} \times \mathbb{R}^{n-n_1-\dots-n_k}$$

Lemma 3.3. $K^j(\mathbb{R}^n / (S_{n_1} \times \dots \times S_{n_k})) = 0, j = 0, 1$, where $S_{n_1} \times \dots \times S_{n_k} \subset S_n$.

Proof. It suffices to prove for $\mathbb{R}^n / (S_{n_1} \times S_{n_2})$. The general case follows by induction on k .

Now, $\mathbb{R}^n / (S_{n_1} \times S_{n_2}) \cong \mathbb{R}^{n_1} / S_{n_1} \times \mathbb{R}^{n-n_1} / S_{n_2}$. Applying the Künneth formula and Lemma 3.2, the result follows. \square

We give now some examples by computing $K_j C_r^* GL(n, \mathbb{R})$ for small n .

Example 3.4. We start with the case of $GL(1, \mathbb{R})$. We have:

$$M = \mathbb{R}^\times, {}^0M = \mathbb{Z}/2\mathbb{Z}, W(M) = 1 \text{ and } X(M) = \mathbb{R}.$$

Hence,

$$\mathcal{A}_1^t(\mathbb{R}) \cong \bigsqcup_{\sigma \in (\mathbb{Z}/2\mathbb{Z})} \mathbb{R}/1 = \mathbb{R} \sqcup \mathbb{R}, \quad (7)$$

and the K -theory is given by

$$K_j C_r^* GL(1, \mathbb{R}) \cong K^j(\mathcal{A}_1^t(\mathbb{R})) = K^j(\mathbb{R} \sqcup \mathbb{R}) = K^j(\mathbb{R}) \oplus K^j(\mathbb{R}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & , j = 1 \\ 0 & , j = 0. \end{cases}$$

Example 3.5. For $GL(2, \mathbb{R})$ we have two partitions of $n = 2$ and the following data

Partition	M	0M	$W(M)$	$X(M)$	$\sigma \in E_2({}^0M)$
$2+0$	$GL(2, \mathbb{R})$	$SL^\pm(2, \mathbb{R})$	1	\mathbb{R}	$\sigma = i_{G,P}(\mathcal{D}_\ell^+), \ell \in \mathbb{N}$
$1+1$	$(\mathbb{R}^\times)^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{R}^2	$\sigma = i_{G,P}(id \otimes sgn)$

Then the tempered dual is parameterized as follows

$$\mathcal{A}_2^t(\mathbb{R}) \cong \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) = \left(\bigsqcup_{\ell \in \mathbb{N}} \mathbb{R} \right) \sqcup (\mathbb{R}^2/S_2) \sqcup (\mathbb{R}^2/S_2) \sqcup \mathbb{R}^2,$$

and the K -theory groups are given by

$$K_j C_r^* GL(2, \mathbb{R}) \cong K^j(\mathcal{A}_2^t(\mathbb{R})) \cong \left(\bigoplus_{\ell \in \mathbb{N}} K^j(\mathbb{R}) \right) \oplus K^j(\mathbb{R}^2) = \begin{cases} \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} & , j = 1 \\ \mathbb{Z} & , j = 0. \end{cases}$$

Example 3.6. For $GL(3, \mathbb{R})$ there are two partitions for $n = 3$, to which correspond the following data

Partition	M	0M	$W(M)$	$X(M)$
$2+1$	$GL(2, \mathbb{R}) \times \mathbb{R}^\times$	$SL^\pm(2, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})$	1	\mathbb{R}^2
$1+1+1$	$(\mathbb{R}^\times)^3$	$(\mathbb{Z}/2\mathbb{Z})^3$	S_3	\mathbb{R}^3

For the partition $3 = 2 + 1$, an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}_\ell^+ \otimes \tau) , \ell \in \mathbb{N} \text{ and } \tau \in \widehat{(\mathbb{Z}/2\mathbb{Z})}.$$

For the partition $3 = 1 + 1 + 1$, an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\bigotimes_{i=1}^3 \tau_i) , \tau_i \in \widehat{(\mathbb{Z}/2\mathbb{Z})}.$$

The tempered dual is parameterized as follows

$$\mathcal{A}_3^t(\mathbb{R}) \cong \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) = \bigsqcup_{\mathbb{N} \times (\mathbb{Z}/2\mathbb{Z})} (\mathbb{R}^2/1) \bigsqcup_{(\mathbb{Z}/2\mathbb{Z})^3} (\mathbb{R}^3/S_3).$$

The K -theory groups are given by

$$K_j C_r^* GL(3, \mathbb{R}) \cong K^j(\mathcal{A}_3^t(\mathbb{R})) \cong \bigoplus_{\mathbb{N} \times (\mathbb{Z}/2\mathbb{Z})} K^j(\mathbb{R}^2) \oplus 0 = \begin{cases} \bigoplus_{\mathbb{N} \times (\mathbb{Z}/2\mathbb{Z})} \mathbb{Z} & , j = 0 \\ 0 & , j = 1. \end{cases}$$

The general case of $GL(n, \mathbb{R})$ will now be considered. It can be split in two cases: n even and n odd.

• $n = 2q$ even

Suppose n is even. For every partition $n = 2q + r$, either $W_\sigma(M) = 1$ or $W_\sigma(M) \neq 1$. If $W_\sigma(M) \neq 1$ then $\mathbb{R}^{n_M}/W_\sigma(M)$ is a cone and the K -groups K^0 and K^1 both vanish. This happens precisely when $r > 2$ and therefore we have only two partitions, corresponding to the choices of $r = 0$ and $r = 2$, which contribute to the K -theory with non-zero K -groups

Partition	M	0M	$W(M)$
$2q$	$GL(2, \mathbb{R})^q$	$SL^\pm(2, \mathbb{R})^q$	S_q
$2(q-1) + 2$	$GL(2, \mathbb{R})^{q-1} \times (\mathbb{R}^\times)^2$	$SL^\pm(2, \mathbb{R})^{q-1} \times (\mathbb{Z}/2\mathbb{Z})^2$	$S_{q-1} \times (\mathbb{Z}/2\mathbb{Z})$

We also have $X(M) \cong \mathbb{R}^q$ for $n = 2q$, and $X(M) \cong \mathbb{R}^{q+1}$, for $n = 2(q-1) + 2$.

For the partition $n = 2q$ ($r = 0$), an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}_{\ell_1}^+ \otimes \dots \otimes \mathcal{D}_{\ell_q}^+) , (\ell_1, \dots, \ell_q) \in \mathbb{N}^q \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$

For the partition $n = 2(q-1) + 2$ ($r = 2$), an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}_{\ell_1}^+ \otimes \dots \otimes \mathcal{D}_{\ell_{q-1}}^+ \otimes id \otimes sgn) , (\ell_1, \dots, \ell_{q-1}) \in \mathbb{N}^{q-1} \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$

Therefore, the tempered dual has the following form

$$\mathcal{A}_n^t(\mathbb{R}) = \mathcal{A}_{2q}^t(\mathbb{R}) = \left(\bigsqcup_{\ell \in \mathbb{N}^q} \mathbb{R}^q \right) \sqcup \left(\bigsqcup_{\ell' \in \mathbb{N}^{q-1}} \mathbb{R}^{q+1} \right) \sqcup \mathcal{C}$$

where \mathcal{C} is a disjoint union of closed cones as in Definition 3.1.

Theorem 3.7. *Suppose $n = 2q$ is even. Then the K -groups are*

$$K_j C_r^* GL(n, \mathbb{R}) \cong \begin{cases} \bigoplus_{\ell \in \mathbb{N}^q} \mathbb{Z} & , j \equiv q \pmod{2} \\ \bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z} & , \text{otherwise.} \end{cases}$$

If $q = 1$ then the direct sum $\bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z}$ will denote a single copy of \mathbb{Z} .

• $n = 2q + 1$ odd

If n is odd only one partition contributes to the K -theory of $GL(n, \mathbb{R})$ with non-zero K -groups:

Partition	M	0M	$W(M)$	$X(M)$
$2q + 1$	$GL(2, \mathbb{R})^{q+1} \times \mathbb{R}^\times$	$SL^\pm(2, \mathbb{R})^q \times (\mathbb{Z}/2\mathbb{Z})$	S_q	\mathbb{R}^{q+1}

An element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}_{\ell_1}^+ \otimes \dots \otimes \mathcal{D}_{\ell_q}^+ \otimes \tau), (\ell_1, \dots, \ell_q, \tau) \in \mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}) \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$

The tempered dual is given by

$$\mathcal{A}_n^t(\mathbb{R}) = \mathcal{A}_{2q+1}^t(\mathbb{R}) = \left(\bigsqcup_{\ell \in (\mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}))} \mathbb{R}^{q+1} \right) \sqcup \mathcal{C}$$

where \mathcal{C} is a disjoint union of closed cones as in Definition 3.1.

Theorem 3.8. *Suppose $n = 2q + 1$ is odd. Then the K -groups are*

$$K_j C_r^* GL(n, \mathbb{R}) \cong \begin{cases} \bigoplus_{\ell \in \mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z})} \mathbb{Z} & , j \equiv q + 1 \pmod{2} \\ 0 & , \text{otherwise.} \end{cases}$$

Here, we use the following convention: if $q = 0$ then the direct sum is $\bigoplus_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$.

We conclude that the K -theory of $C_r^* GL(n, \mathbb{R})$ depends on essentially one parameter q given by the maximum number of 2's in the partitions of n into 1's and 2's. If n is even then $q = \frac{n}{2}$ and if n is odd then $q = \frac{n-1}{2}$.

3.2 K -theory for $GL(n, \mathbb{C})$

Let T° be the maximal compact subgroup of the maximal compact torus T of $GL(n, \mathbb{C})$. Let σ be a unitary character of T° . We note that $W = W(T)$, $W_\sigma = W_\sigma(T)$. If $W_\sigma = 1$ then we say that the orbit $W \cdot \sigma$ is *generic*.

Theorem 3.9. *The K -theory of $C_r^* GL(n, \mathbb{C})$ admits the following description. If $n = j \pmod{2}$ then K_j is free abelian on countably many generators, one for each generic W -orbit in the unitary dual of T° , and $K_{j+1} = 0$.*

Proof. We exploit the strong Morita equivalence described in [13, Prop. 4.1]. We have a homeomorphism of locally compact Hausdorff spaces:

$$\mathcal{A}_n^t(\mathbb{C}) \cong \bigsqcup X(T)/W_\sigma(T)$$

by the Harish-Chandra Plancherel Theorem for complex reductive groups [6], and the identification of the Jacobson topology on the left-hand-side with the natural topology on the right-hand-side, as in [13]. The result now follows from Lemma 4.3. \square

4 Langlands parameters for $GL(n)$

The Weil group of \mathbb{C} is simply

$$W_{\mathbb{C}} \cong \mathbb{C}^{\times},$$

and the Weil group of \mathbb{R} can be written as disjoint union

$$W_{\mathbb{R}} \cong \mathbb{C}^{\times} \sqcup j\mathbb{C}^{\times},$$

where $j^2 = -1$ and $jcj^{-1} = \bar{c}$ (\bar{c} denotes complex conjugation). We shall use this disjoint union to describe the representation theory of $W_{\mathbb{R}}$.

Definition 4.1. *An L -parameter is a continuous homomorphism*

$$\phi : W_F \rightarrow GL(n, \mathbb{C})$$

such that $\phi(w)$ is semisimple for all $w \in W_F$.

L -parameters are also called Langlands parameters. Two L -parameters are equivalent if they are conjugate under $GL(n, \mathbb{C})$. The set of equivalence classes of L -parameters is denoted by \mathcal{G}_n . And the set of equivalence classes of L -parameters whose image is bounded is denoted by \mathcal{G}_n^t .

Let F be either \mathbb{R} or \mathbb{C} . Let $\mathcal{A}_n(F)$ (resp. $\mathcal{A}_n^t(F)$) denote the smooth dual (resp. tempered dual) of $GL(n, F)$. The local Langlands correspondence is a bijection

$$\mathcal{G}_n(F) \rightarrow \mathcal{A}_n(F).$$

In particular,

$$\mathcal{G}_n^t(F) \rightarrow \mathcal{A}_n^t(F)$$

is also a bijection.

We are only interested in L -parameters whose image is bounded. In the sequel we will refer to them, for simplicity, as L -parameters.

L -parameters for $W_{\mathbb{C}}$

A 1-dimensional L -parameter for $W_{\mathbb{C}}$ is simply a character of \mathbb{C}^{\times} (i.e. a unitary quasicharacter):

$$\chi(z) = \left(\frac{z}{|z|}\right)^{\ell} \otimes |z|^{it}$$

where $|z| = |z|_{\mathbb{C}} = z\bar{z}$, $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$. To emphasize the dependence on parameters (ℓ, t) we write sometimes $\chi = \chi_{\ell, t}$.

An n -dimensional L -parameter can be written as a direct sum of n 1-dimensional characters of \mathbb{C}^\times :

$$\phi = \phi_1 \oplus \dots \oplus \phi_n,$$

with $\phi_k(z) = (\frac{z}{|z|})^{\ell_k} \otimes |z|^{t_k}$, $\ell_k \in \mathbb{Z}$, $t_k \in \mathbb{R}$, $k = 1, \dots, n$.

L -parameters for $W_{\mathbb{R}}$

The 1-dimensional L -parameters for $W_{\mathbb{R}}$ are as follows

$$\begin{cases} \phi_{\varepsilon,t}(z) = |z|_{\mathbb{R}}^{it} \\ \phi_{\varepsilon,t}(j) = (-1)^{\varepsilon} \end{cases}, \varepsilon \in \{0, 1\}, t \in \mathbb{R}.$$

We may now describe the local Langlands correspondence for $GL(1, \mathbb{R})$:

$$\begin{aligned} \phi_{0,t} &\mapsto 1 \otimes |\cdot|_{\mathbb{R}}^{it} \\ \phi_{1,t} &\mapsto sgn \otimes |\cdot|_{\mathbb{R}}^{it} \end{aligned}$$

Now, we consider 2-dimensional L -parameters for $W_{\mathbb{R}}$:

$$\phi_{\ell,t}(z) = \begin{pmatrix} \chi_{\ell,t}(z) & 0 \\ 0 & \bar{\chi}_{\ell,t}(z) \end{pmatrix}, \phi_{\ell,t}(j) = \begin{pmatrix} 0 & (-1)^{\ell} \\ 1 & 0 \end{pmatrix}.$$

with $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$.

and

$$\phi_{m,t,n,s}(z) = \begin{pmatrix} \chi_{0,t}(z) & 0 \\ 0 & \chi_{0,s}(z) \end{pmatrix}, \phi_{m,t,n,s}(j) = \begin{pmatrix} (-1)^m & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

with $m, n \in \{0, 1\}$ and $t, s \in \mathbb{R}$.

The local Langlands correspondence for $GL(2, \mathbb{R})$ may be described as follows.

The L -parameter ϕ_{m_1,t_1,m_2,t_2} corresponds, via Langlands correspondence, to the unitary principal series:

$$\phi_{m_1,t_1,m_2,t_2} \mapsto \pi(\mu_1, \mu_2),$$

where μ_i is the character of \mathbb{R}^\times given by

$$\mu_i(x) = (\frac{x}{|x|})^{m_i} |x|^{it_i}, m_i \in \{0, 1\}, t_i \in \mathbb{R}.$$

The L -parameter $\phi_{\ell,t}$ corresponds, via the Langlands correspondence, to the discrete series:

$$\phi_{\ell,t} \mapsto D_\ell \otimes |\det(\cdot)|_{\mathbb{R}}^{it}, \text{ with } \ell \in \mathbb{N}, t \in \mathbb{R}.$$

Proposition 4.2. (i) $\phi_{\ell,t} \cong \phi_{-\ell,t}$;

$$(ii) \quad \phi_{\ell,t,m,s} \cong \phi_{m,s,\ell,t};$$

$$(iii) \quad \phi_{0,t} \cong \phi_{1,t,0,t} \cong \phi_{0,t,1,t};$$

The proof is elementary. We now quote the following result.

Lemma 4.3. [8] *Every finite-dimensional semi-simple representation ϕ of $W_{\mathbb{R}}$ is fully reducible, and each irreducible representation has dimension one or two.*

5 Base change

We may state the base change problem for archimedean fields in the following way. Consider the archimedean base change \mathbb{C}/\mathbb{R} . We have $W_{\mathbb{C}} \subset W_{\mathbb{R}}$ and there is a natural map

$$Res_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} : \mathcal{G}_n(\mathbb{R}) \longrightarrow \mathcal{G}_n(\mathbb{C}) \quad (8)$$

called *restriction*. By the local Langlands correspondence for archimedean fields [3, Theorem 3.1, p.236][8], there is a base change map

$$\mathcal{BC} : \mathcal{A}_n(\mathbb{R}) \longrightarrow \mathcal{A}_n(\mathbb{C}) \quad (9)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}_n(\mathbb{R}) & \xrightarrow{\mathcal{BC}} & \mathcal{A}_n(\mathbb{C}) \\ \uparrow \mathbb{R}\mathcal{L}_n & & \uparrow \mathbb{C}\mathcal{L}_n \\ \mathcal{G}_n(\mathbb{R}) & \xrightarrow[Res_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}]{} & \mathcal{G}_n(\mathbb{C}) \end{array}$$

Arthur and Clozel's book [1] gives a full treatment of base change for $GL(n)$. The case of archimedean base change can be captured in an elegant formula [1, p. 71]. We briefly review these results.

Given a partition $n = 2q + r$ let χ_i ($i = 1, \dots, q$) be a ramified character of \mathbb{C}^\times and let ξ_j ($j = 1, \dots, r$) be a ramified character of \mathbb{R}^\times . Since the χ_i 's

are ramified, $\chi_i(z) \neq \chi_i^\tau(z) = \chi_i(\bar{z})$, where τ is a generator of $Gal(\mathbb{C}/\mathbb{R})$. By Langlands classification [8], each χ_i defines a discrete series representation $\pi(\chi_i)$ of $GL(2, \mathbb{R})$, with $\pi(\chi_i) = \pi(\chi_i^\tau)$. Denote by $\pi(\chi_1, \dots, \chi_q, \xi_1, \dots, \xi_r)$ the *generalized principal series representation* of $GL(n, \mathbb{R})$

$$\pi(\chi_1, \dots, \chi_q, \xi_1, \dots, \xi_r) = i_{GL(n, \mathbb{R}), MN}(\pi(\chi_1) \otimes \dots \otimes \pi(\chi_q) \otimes \xi_1 \otimes \dots \otimes \xi_r \otimes 1). \quad (10)$$

The base change map for the general principal series representation is given by induction from the Borel subgroup $B(\mathbb{C})$ [1, p. 71]:

$$\mathcal{BC}(\pi) = \Pi(\chi_1, \dots, \chi_q, \xi_1, \dots, \xi_r) = i_{GL(n, \mathbb{C}), B(\mathbb{C})}(\chi_1, \chi_1^\tau, \dots, \chi_q, \chi_q^\tau, \xi_1 \circ N, \dots, \xi_r \circ N), \quad (11)$$

where $N = N_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$ is the norm map defined by $z \mapsto z\bar{z}$.

We illustrate the base change map with two simple examples.

Example 5.1. For $n = 1$, base change is simply composition with the norm map

$$\mathcal{BC} : \mathcal{A}_1^t(\mathbb{R}) \rightarrow \mathcal{A}_1^t(\mathbb{C}), \quad \mathcal{BC}(\chi) = \chi \circ N.$$

Example 5.2. For $n = 2$, there are two different kinds of representations, one for each partition of 2. According to (10), $\pi(\chi)$ corresponds to the partition $2 = 2 + 0$ and $\pi(\xi_1, \xi_2)$ corresponds to the partition $2 = 1 + 1$. Then the base change map is given, respectively, by

$$\mathcal{BC}(\pi(\chi)) = i_{GL(2, \mathbb{C}), B(\mathbb{C})}(\chi, \chi^\tau),$$

and

$$\mathcal{BC}(\pi(\xi_1, \xi_2)) = i_{GL(2, \mathbb{C}), B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N).$$

5.1 The base change map

Now, we define base change as a map of topological spaces and study the induced K -theory map.

Proposition 5.3. The base change map $\mathcal{BC} : \mathcal{A}_n^t(\mathbb{R}) \rightarrow \mathcal{A}_n^t(\mathbb{C})$ is a continuous proper map.

Proof. First, we consider the case $n = 1$. As we have seen in Example 5.1, base change for $GL(1)$ is the map given by $\mathcal{BC}(\chi) = \chi \circ N$, for all characters $\chi \in \mathcal{A}_1^t(\mathbb{R})$, where $N : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$ is the norm map.

Let $z \in \mathbb{C}^\times$. We have

$$\mathcal{BC}(\chi)(z) = \chi(|z|^2) = |z|^{2it}. \quad (12)$$

A generic element from $\mathcal{A}_1^t(\mathbb{C})$ has the form

$$\mu(z) = \left(\frac{z}{|z|}\right)^\ell |z|^{it}, \quad (13)$$

where $\ell \in \mathbb{Z}$ and $t \in S^1$, as stated before. Viewing the Pontryagin duals $\mathcal{A}_1^t(\mathbb{R})$ and $\mathcal{A}_1^t(\mathbb{C})$ as topological spaces by forgetting the group structure, and comparing (12) and (13), the base change map can be defined as the following continuous map

$$\begin{aligned} \varphi : \mathcal{A}_1^t(\mathbb{R}) \cong \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) &\longrightarrow \mathcal{A}_1^t(\mathbb{C}) \cong \mathbb{R} \times \mathbb{Z} \\ \chi = (t, \varepsilon) &\mapsto (2t, 0) \end{aligned}$$

A compact subset of $\mathbb{R} \times \mathbb{Z}$ in the connected component $\{\ell\}$ of \mathbb{Z} has the form $K \times \{\ell\} \subset \mathbb{R} \times \mathbb{Z}$, where $K \subset \mathbb{R}$ is compact. We have

$$\varphi^{-1}(K \times \{\ell\}) = \begin{cases} \emptyset & , \text{ if } \ell \neq 0 \\ \frac{1}{2}K \times \{\varepsilon\} & , \text{ if } \ell = 0, \end{cases}$$

where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Therefore $\varphi^{-1}(K \times \{\ell\})$ is compact and φ is proper.

The Case $n > 1$. Base change determines a map $\mathcal{BC} : \mathcal{A}_n^t(\mathbb{R}) \rightarrow \mathcal{A}_n^t(\mathbb{C})$ of topological spaces. Let $X = X(M)/W_\sigma(M)$ be a connected component of $\mathcal{A}_n^t(\mathbb{R})$. Then, X is mapped under \mathcal{BC} into a connected component $Y = Y(T)/W_{\sigma'}(T)$ of $\mathcal{A}_n^t(\mathbb{C})$. Given a generalized principal series representation

$$\pi(\chi_1, \dots, \chi_q, \xi_1, \dots, \xi_r)$$

where the χ_i 's are ramified characters of \mathbb{C}^\times and the ξ 's are ramified characters of \mathbb{R}^\times , then

$$\mathcal{BC}(\pi) = i_{G,B}(\chi_1, \chi_1^\tau, \dots, \chi_q, \chi_q^\tau, \xi_1 \circ N, \dots, \xi_r \circ N).$$

Here, $N = N_{\mathbb{C}/\mathbb{R}}$ is the norm map and τ is the generator of $\text{Gal}(\mathbb{C}/\mathbb{R})$.

We associate to π the usual parameters uniquely defined for each character χ and ξ . For simplicity, we write the set of parameters as a $(q+r)$ -uple:

$$(t, t') = (t_1, \dots, t_q, t'_1, \dots, t'_r) \in \mathbb{R}^{q+r} \cong X(M).$$

Now, if $\pi(\chi_1, \dots, \chi_q, \xi_1, \dots, \xi_r)$ lies in the connected component defined by the fixed parameters $(\ell, \varepsilon) \in \mathbb{Z}^q \times (\mathbb{Z}/2\mathbb{Z})^r$, then

$$(t, t') \in X(M) \mapsto (t, t, 2t') \in Y(T)$$

is a continuous proper map.

It follows that

$$\mathcal{BC} : X(M)/W_\sigma(M) \rightarrow Y(T)/W_{\sigma'}(T)$$

is continuous and proper since the orbit spaces are endowed with the quotient topology. \square

Theorem 5.4. *The functorial map induced by base change*

$$K_j(C_r^*GL(n, \mathbb{C})) \xrightarrow{K_j(\mathcal{BC})} K_j(C_r^*GL(n, \mathbb{R}))$$

is zero for $n > 1$.

Proof. We start with the case $n > 2$. Let $n = 2q + r$ be a partition and M the associated Levi subgroup of $GL(n, \mathbb{R})$. Denote by $X_{\mathbb{R}}(M)$ the unramified characters of M . As we have seen, $X_{\mathbb{R}}(M)$ is parametrized by \mathbb{R}^{q+r} . On the other hand, the only Levi subgroup of $GL(n, \mathbb{C})$ for $n = 2q + r$ is the diagonal subgroup $X_{\mathbb{C}}(M) = (\mathbb{C}^\times)^{2q+r}$.

If $q = 0$ then $r = n$ and both $X_{\mathbb{R}}(M)$ and $X_{\mathbb{C}}(M)$ are parametrized by \mathbb{R}^n . But then in the real case an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{GL(n, \mathbb{R}), P}(\chi_1 \otimes \dots \otimes \chi_n),$$

with $\chi_i \in \widehat{\mathbb{Z}/2\mathbb{Z}}$. Since $n \geq 3$ there is always repetition of the χ_i 's. It follows that the isotropy subgroups $W_\sigma(M)$ are all nontrivial and the quotient spaces \mathbb{R}^n/W_σ are closed cones. Therefore, the K -theory groups vanish.

If $q \neq 0$, then $X_{\mathbb{R}}(M)$ is parametrized by \mathbb{R}^{q+r} and $X_{\mathbb{C}}(M)$ is parametrized by \mathbb{R}^{2q+r} (see Propositions 2.2 and 2.3).

Base change creates a map

$$\mathbb{R}^{q+r} \longrightarrow \mathbb{R}^{2q+r}.$$

Composing with the stereographic projections we obtain a map

$$S^{q+r} \longrightarrow S^{2q+r}$$

between spheres. Any such map is nullhomotopic [4, Proposition 17.9]. Therefore, the induced K -theory map

$$K^j(S^{2q+r}) \longrightarrow K^j(S^{q+r})$$

is the zero map.

The Case $n = 2$. For $n = 2$ there are two Levi subgroups of $GL(2, \mathbb{R})$, $M_1 \cong GL(2, \mathbb{R})$ and the diagonal subgroup $M_2 \cong (\mathbb{R}^\times)^2$. By Proposition 2.2

$X(M_1)$ is parametrized by \mathbb{R} and $X(M_2)$ is parametrized by \mathbb{R}^2 . The group $GL(2, \mathbb{C})$ has only one Levi subgroup, the diagonal subgroup $M \cong (\mathbb{C}^\times)^2$. From Proposition 2.3 it is parametrized by \mathbb{R}^2 .

Since $K^1(\mathcal{A}_2^t(\mathbb{C})) = 0$ by Theorem 5.1, we only have to consider the K^0 functor. The only contribution to $K^0(\mathcal{A}_2^t(\mathbb{R}))$ comes from $M_2 \cong (\mathbb{R}^\times)^2$ and we have (see Example 3.5)

$$K^0(\mathcal{A}_2^t(\mathbb{R})) \cong \mathbb{Z}.$$

For the Levi subgroup $M_2 \cong (\mathbb{R}^\times)^2$, base change is

$$\begin{aligned} \mathcal{BC} : \mathcal{A}_2^t(\mathbb{R}) &\longrightarrow \mathcal{A}_2^t(\mathbb{C}) \\ \pi(\xi_1, \xi_2) &\mapsto i_{GL(2, \mathbb{C}), B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N), \end{aligned}$$

Therefore, it maps a class $[t_1, t_2]$, which lies in the connected component $(\varepsilon_1, \varepsilon_2)$, into the class $[2t_1, 2t_2]$, which lies in the connect component $(0, 0)$. In other words, base change maps a generalized principal series $\pi(\xi_1, \xi_2)$ into a nongeneric point of $\mathcal{A}_2^t(\mathbb{C})$. It follows from Theorem 3.9 that

$$K^0(\mathcal{BC}) : K^0(\mathcal{A}_2^t(\mathbb{R})) \rightarrow K^0(\mathcal{A}_2^t(\mathbb{C}))$$

is the zero map. □

5.2 Base change in one dimension

In this section we consider base change for $GL(1)$.

Theorem 5.5. *The functorial map induced by base change*

$$K_1(C_r^*GL(1, \mathbb{C})) \xrightarrow{K_1(\mathcal{BC})} K_1(C_r^*GL(1, \mathbb{R}))$$

is given by $K_1(\mathcal{BC}) = \Delta \circ Pr$, where Pr is the projection of the zero component of $K^1(\mathcal{A}_1^t(\mathbb{C}))$ into \mathbb{Z} and Δ is the diagonal $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$.

Proof. For $GL(1)$, base change

$$\chi \in \mathcal{A}_1^t(\mathbb{R}) \mapsto \chi \circ N_{\mathbb{C}/\mathbb{R}} \in \mathcal{A}_1^t(\mathbb{C})$$

induces a map

$$K^1(\mathcal{BC}) : K^1(\mathcal{A}_1^t(\mathbb{C})) \rightarrow K^1(\mathcal{A}_1^t(\mathbb{R})).$$

Any character $\chi \in \mathcal{A}_1^t(\mathbb{R})$ is uniquely determined by a pair of parameters $(t, \varepsilon) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. Similarly, any character $\mu \in \mathcal{A}_1^t(\mathbb{C})$ is uniquely determined by a pair of parameters $(t, \ell) \in \mathbb{R} \times \mathbb{Z}$. The discrete parameter ε (resp., ℓ) labels each connected component of $\mathcal{A}_1^t(\mathbb{R}) = \mathbb{R} \sqcup \mathbb{R}$ (resp., $\mathcal{A}_1^t(\mathbb{C}) = \bigsqcup_{\mathbb{Z}} \mathbb{R}$).

Base change maps each component ε of $\mathcal{A}_1^t(\mathbb{R})$ into the component 0 of $\mathcal{A}_1^t(\mathbb{C})$, sending $t \in \mathbb{R}$ to $2t \in \mathbb{R}$. The map $t \mapsto 2t$ is homotopic to the identity. At the level of K^1 , the base change map is given by $K_1(BC) = \Delta \circ Pr$, where Pr is the projection of the zero component of $K^1(\mathcal{A}_1^t(\mathbb{C}))$ into \mathbb{Z} and Δ is the diagonal $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$.

□

6 Automorphic induction

We begin this section by describing the action of $Gal(\mathbb{C}/\mathbb{R})$ on $\widehat{W_{\mathbb{C}}} = \widehat{\mathbb{C}^{\times}}$. Take $\chi = \chi_{\ell,t} \in \widehat{\mathbb{C}^{\times}}$ and let τ denote the nontrivial element of $Gal(\mathbb{C}/\mathbb{R})$. Then, $Gal(\mathbb{C}/\mathbb{R})$ acts on $\widehat{\mathbb{C}^{\times}}$ as follows:

$$\chi^{\tau}(z) = \chi(\bar{z}).$$

Hence,

$$\chi_{\ell,t}^{\tau}(z) = \left(\frac{\bar{z}}{|z|}\right)^{\ell} |z|_{\mathbb{C}}^{it} = \left(\frac{z}{|z|}\right)^{-\ell} |z|_{\mathbb{C}}^{it}$$

and we conclude that

$$\chi_{\ell,t}^{\tau}(z) = \chi_{-\ell,t}(z).$$

In particular,

$$\chi^{\tau} = \chi \Leftrightarrow \ell = 0 \Leftrightarrow \chi = |\cdot|_{\mathbb{C}}^{it}$$

i.e, χ is unramified.

Note that $W_{\mathbb{C}} \subset W_{\mathbb{R}}$, with index $[W_{\mathbb{R}} : W_{\mathbb{C}}] = 2$. Therefore, there is a natural induction map

$$Ind_{\mathbb{C}/\mathbb{R}} : \mathcal{G}_1^t(\mathbb{C}) \rightarrow \mathcal{G}_2^t(\mathbb{R}).$$

By the local Langlands correspondence for archimedean fields [3, 8], there exists an automorphic induction map $\mathcal{AI}_{\mathbb{C}/\mathbb{R}}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}_1^t(\mathbb{C}) & \xrightarrow{\mathcal{AI}_{\mathbb{C}/\mathbb{R}}} & \mathcal{A}_2^t(\mathbb{R}) \\ \uparrow \mathbb{C}\mathcal{L}_1 & & \uparrow \mathbb{R}\mathcal{L}_2 \\ \mathcal{G}_1^t(\mathbb{C}) & \xrightarrow{Ind_{\mathbb{C}/\mathbb{R}}} & \mathcal{G}_2^t(\mathbb{R}) \end{array}$$

The next result describes reducibility of induced representations.

Proposition 6.1. *Let χ be a character of $W_{\mathbb{C}}$. We have:*

- (i) *If $\chi \neq \chi^\tau$ then $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$ is irreducible;*
- (ii) *If $\chi = \chi^\tau$ then $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$ is reducible. Moreover, there exist $\rho \in \widehat{W_{\mathbb{R}}}$ such that*

$$Ind_{\mathbb{C}/\mathbb{R}}(\chi) = \rho \oplus \rho^\tau = \rho \oplus sgn.\rho,$$

where $\rho|_{W_{\mathbb{C}}} = \chi$;

- (iii) *$Ind_{\mathbb{C}/\mathbb{R}}(\chi_1) \cong Ind_{\mathbb{C}/\mathbb{R}}(\chi_2)$ if and only if $\chi_1 = \chi_2$ or $\chi_1 = \chi_2^\tau$.*

Proof. Apply Frobenius reciprocity

$$Hom_{W_{\mathbb{R}}}(Ind_{\mathbb{C}/\mathbb{R}}(\chi_1), Ind_{\mathbb{C}/\mathbb{R}}(\chi_2)) \cong Hom_{W_{\mathbb{C}}}(\chi_1, \chi_2).$$

Now, $W_{\mathbb{R}} = W_{\mathbb{C}} \sqcup jW_{\mathbb{C}}$. Therefore, the restriction of $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$ to $W_{\mathbb{C}}$ is $\chi \oplus \chi^\tau$. The result follows since $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$ is semi-simple. \square

Proposition 6.2. *A finite dimensional continuous irreducible representation of $W_{\mathbb{R}}$ is either a character or isomorphic to some $Ind_{\mathbb{C}/\mathbb{R}}(\chi)$, with $\chi \neq \chi^\tau$.*

Proof. It follows immediately from Lemma 4.3 \square

6.1 The automorphic induction map

In this section we describe automorphic induction map as a map of topological spaces. We begin by considering $n = 1$.

Let $\chi = \chi_{\ell,t}$ be a character of $W_{\mathbb{C}}$. If $\chi \neq \chi^\tau$, by proposition 4.2, $\phi_{\ell,t} \simeq \phi_{-\ell,t}$. Hence,

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}\mathcal{L}_1(\chi_{\ell,t})) = D_{|\ell|} \otimes |det(\cdot)|^{it}.$$

On the other hand, if $\chi = \chi^\tau$ then $\chi = \chi_{0,t}$ and $\chi(z) = |z|_{\mathbb{C}}^{it}$. Therefore,

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}\mathcal{L}_1(|\cdot|_{\mathbb{C}}^{it})) = \mathbb{R}\mathcal{L}_2(\rho \oplus sgn.\rho) = \pi(\rho, \rho^{-1}),$$

where $\pi(\rho, \rho^{-1})$ is a reducible principal series and ρ is the character of $\mathbb{R}^\times \simeq W_{\mathbb{R}}^{ab}$ associated with $\chi_{0,t} = |\cdot|_{\mathbb{C}}^{it}$ via class field theory, i.e. $\rho|_{W_{\mathbb{C}}} = \chi$.

Recall that

$$\mathcal{A}_1^t(\mathbb{C}) \cong \bigsqcup_{\ell \in \mathbb{Z}} \mathbb{R}$$

and

$$\mathcal{A}_2^t(\mathbb{R}) \cong \left(\bigsqcup_{\ell \in \mathbb{N}} \mathbb{R} \right) \bigsqcup (\mathbb{R}^2/S_2) \bigsqcup (\mathbb{R}^2/S_2) \bigsqcup \mathbb{R}^2.$$

As a map of topological spaces, automorphic induction for $n = 1$ may be described as follows:

$$(t, \ell) \in \mathbb{R} \times \mathbb{Z} \mapsto (t, |\ell|) \in \mathbb{R} \times \mathbb{N}, \text{ if } \ell \neq 0 \quad (14)$$

$$(t, 0) \in \mathbb{R} \times \mathbb{Z} \mapsto (t, t) \mapsto \mathbb{R}^2, \text{ if } \ell = 0. \quad (15)$$

More generally, let $\chi_1 \oplus \dots \oplus \chi_n$ be an n dimensional L -parameter of $W_{\mathbb{C}}$. Then, either $\chi_k \neq \chi_k^{\tau}$ for every k , in which case automorphic induction is

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}\mathcal{L}_n(\chi_1 \oplus \dots \oplus \chi_n)) = D_{|\ell_1|} \otimes |\det(\cdot)|^{it_1} \oplus \dots \oplus D_{|\ell_n|} \otimes |\det(\cdot)|^{it_n} \quad (16)$$

or for some k (possibly more than one), $\chi_k = \chi_k^{\tau}$, in which case we have

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}\mathcal{L}_n(\chi_1 \oplus \dots \oplus |\cdot|_{\mathbb{C}}^{it_k} \oplus \dots \oplus \chi_n)) = D_{|\ell_1|} \otimes |\det(\cdot)|^{it_1} \oplus \dots \quad (17)$$

$$\oplus \pi(\rho_k, \rho_k^{-1}) \oplus \dots \oplus D_{|\ell_n|} \otimes |\det(\cdot)|^{it_n}.$$

In order to describe automorphic induction as a map of topological spaces, it is enough to consider components of $\mathcal{A}_n^t(\mathbb{C})$ with generic W -orbit. For convenience, we introduce the following notation:

if (t_1, \dots, t_n) is in the component of $\mathcal{A}_n^t(\mathbb{C})$ labeled by (ℓ_1, \dots, ℓ_n) , i.e

$$(t_1, \dots, t_n) \in (\mathbb{R} \times \{\ell_1\}) \times \dots \times (\mathbb{R} \times \{\ell_n\})$$

we write simply

$$(t_1, \dots, t_n) \in \mathbb{R}_{(\ell_1, \dots, \ell_n)}^n, \quad \ell_i \in \mathbb{Z}.$$

There are two cases:

Case 1: $\chi_k \neq \chi_k^{\tau}$, i.e. $\ell_k \neq 0$, for every k ,

$$\mathcal{AI} : (t_1, \dots, t_n) \in \mathbb{R}_{(\ell_1, \dots, \ell_n)}^n \mapsto (t_1, \dots, t_n) \in \mathbb{R}_{(|\ell_1|, \dots, |\ell_n|)}^n \quad (18)$$

So, $(|\ell_1|, \dots, |\ell_n|) \in \mathbb{N}^n$.

Case 2: if there are $0 < m < n$ characters such that $\chi_k = \chi_k^{\tau}$, then

$$\mathcal{AI} : (t_1, \dots, t_k, \dots, t_n) \in \mathbb{R}_{(\ell_1, \dots, 0, \dots, \ell_n)}^n \mapsto (t_1, \dots, t_k, t_k, \dots, t_n) \in (\mathbb{R}^n/W)_{(|\ell_1|, \dots, |\ell_n|)^*} \quad (19)$$

where $(|\ell_1|, \dots, |\ell_n|)^* \in \mathbb{N}^{n-m}$ means that we have deleted the m labels corresponding to $\ell_k = 0$. Note that if $m > 1$, necessarily $W \neq 0$.

We have the following result

Proposition 6.3. *The automorphic induction map*

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{C}) \rightarrow \mathcal{A}_{2n}^t(\mathbb{R})$$

is a continuous proper map.

The proof follows from the above discussion and is similar to that of proposition 5.3.

Example 6.4. Consider $n = 3$. Then,

$$\mathcal{A}_3^t(\mathbb{C}) \simeq \bigsqcup_{\sigma} \mathbb{R}^3 / W_{\sigma}$$

and

$$\mathcal{A}_6^t(\mathbb{R}) \simeq \left(\bigsqcup_{\ell \in \mathbb{N}^3} \mathbb{R}^3 \right) \sqcup \left(\bigsqcup_{\ell' \in \mathbb{N}^2} \mathbb{R}^4 \right) \sqcup \mathcal{C},$$

where \mathcal{C} is a disjoint union of cones.

Let $\chi_1 \oplus \chi_2 \oplus \chi_3$ denote a 3-dimensional L -parameter of $W_{\mathbb{C}}$. We have the following description of $\mathcal{AI}_{\mathbb{C}/\mathbb{R}}$ as a map of topological spaces:

- $\chi_1 \neq \chi_1^{\tau}, \chi_2 \neq \chi_2^{\tau}, \chi_3 \neq \chi_3^{\tau}$

$$(t_1, t_2, t_3) \in \mathbb{R}_{(\ell_1, \ell_2, \ell_3)}^3 \longmapsto (t_1, t_2, t_3) \in \mathbb{R}_{(|\ell_1|, |\ell_2|, |\ell_3|)}^3$$

with $\ell_i \in \mathbb{Z} \setminus \{0\}$.

- $\chi_1 = \chi_1^{\tau}, \chi_2 \neq \chi_2^{\tau}, \chi_3 \neq \chi_3^{\tau}$

$$(t_1, t_2, t_3) \in \mathbb{R}_{(0, \ell_2, \ell_3)}^3 \longmapsto (t_1, t_1, t_2, t_3) \in (\mathbb{R}^4 / W)_{(|\ell_2|, |\ell_3|)}$$

with $\ell_i \in \mathbb{Z} \setminus \{0\}$. Similar for the cases $(\ell_1, 0, \ell_3)$ and $(\ell_1, \ell_2, 0)$.

- $\chi_1 = \chi_1^{\tau}, \chi_2 = \chi_2^{\tau}, \chi_3 \neq \chi_3^{\tau}$

$$(t_1, t_2, t_3) \in \mathbb{R}_{(0, 0, \ell_3)}^3 \longmapsto (t_1, t_1, t_2, t_2, t_3) \in (\mathbb{R}^5 / W)_{(|\ell_3|)}$$

with $\ell_3 \in \mathbb{Z} \setminus \{0\}$. Similar for the cases $(\ell_1, 0, 0)$ and $(0, \ell_2, 0)$.

- $\chi_1 = \chi_1^{\tau}, \chi_2 = \chi_2^{\tau}, \chi_3 = \chi_3^{\tau}$

$$(t_1, t_2, t_3) \in \mathbb{R}_{(0, 0, 0)}^3 \longmapsto (t_1, t_1, t_2, t_2, t_3, t_3) \in (\mathbb{R}^6 / W)$$

6.2 Automorphic induction in one dimension

Automorphic induction \mathcal{AI} induces a K -theory map at the level of K -theory groups K^1 :

$$K^1(\mathcal{AI}) : K^1(\mathcal{A}_2^t(\mathbb{R})) \rightarrow K^1(\mathcal{A}_1^t(\mathbb{C})). \quad (20)$$

We have

$$K^1(\mathcal{A}_2^t(\mathbb{R})) \cong \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z}.$$

Each class of 1-dimension L -parameters of $W_{\mathbb{C}}$ (characters of \mathbb{C}^\times)

$$[\chi] = [\chi_{\ell,t}] = [\chi_{-\ell,t}] \quad (\ell \neq 0)$$

contributes with one generator to $K^1(\mathcal{A}_2^t(\mathbb{R}))$. Note that, under \mathcal{AI} , this is precisely the parametrization given by the discrete series $D_{|\ell|}$.

On the other hand,

$$K^1(\mathcal{A}_1^t(\mathbb{C})) \cong \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}.$$

Again, each class (of characters of \mathbb{C}^\times) $[\chi]$ contributes with a generator to $K^1(\mathcal{A}_1^t(\mathbb{C}))$, only this time $[\chi_{\ell,t}] \neq [\chi_{-\ell,t}]$, i.e. ℓ and $-\ell$ belong to different classes.

Note that we may write

$$K^1(\mathcal{A}_2^t(\mathbb{R})) \cong \bigoplus_{\text{Discrete series}} \mathbb{Z} = \bigoplus_{[D_{|\ell|}]} \mathbb{Z} = \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z}$$

and

$$K^1(\mathcal{A}_1^t(\mathbb{C})) \cong \bigoplus_{\widehat{\mathbb{C}^\times}} \mathbb{Z} = \bigoplus_{[\chi_\ell]} \mathbb{Z} = \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}.$$

The automorphic induction map

$$K^1(\mathcal{AI}) : K^1(\mathcal{A}_2^t(\mathbb{R})) \rightarrow K^1(\mathcal{A}_1^t(\mathbb{C}))$$

may be interpreted, at the level of K^1 , as a kind of "shift" map

$$[D_{|\ell|}] \mapsto [\chi_{|\ell|}]$$

More explicitly, the "shift" map is

$$\bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} \rightarrow \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}, ([D_1], [D_2], \dots) \mapsto (\dots, 0, 0, [\chi_1], [\chi_2], \dots)$$

where the image under $K^1(\mathcal{AI})$ on each component of the right hand side with label $\ell \leq 0$ is zero (because $K^1(\mathcal{AI})$ is a group homomorphism so it must map zero into zero).

6.3 Automorphic induction in n dimensions

In this section we consider automorphic induction for $GL(n)$. Contrary to base change (see theorems 5.4 and 5.5), the K -theory map of automorphic induction is nonzero for every n .

Theorem 6.5. *The functorial map induced by automorphic induction*

$$K_j(C_r^*GL(2n, \mathbb{R})) \xrightarrow{K_j(\mathcal{AI})} K_j(C_r^*GL(n, \mathbb{C}))$$

is given by

$$[D_{|\ell_1|} \otimes \dots \otimes D_{|\ell_n|}] \mapsto [\chi_{|\ell_1|} \oplus \dots \oplus \chi_{|\ell_n|}]$$

if $n \equiv j \pmod{2}$ and $\chi_k \neq \chi_k^\tau$ for every k , and is zero otherwise.

Here, $[D_{|\ell_1|} \otimes \dots \otimes D_{|\ell_n|}]$ denotes the generator of the component $\mathbb{Z}_{(|\ell_1|, \dots, |\ell_n|)}$ of $K_j(C_r^*GL(2n, \mathbb{R}))$ and $[\chi_{|\ell_1|} \oplus \dots \oplus \chi_{|\ell_n|}]$ is the generator of the component $\mathbb{Z}_{(|\ell_1|, \dots, |\ell_n|)}$ of $K_j(C_r^*GL(n, \mathbb{C}))$

Proof. Let $0 \leq m < n$ be the number of characters χ_k with $\chi_k = \chi_k^\tau$.

Case 1: $m = 0$

In this case, $\chi_k \neq \chi_k^\tau$ for every k . Each character χ_{ℓ_k} , $\ell_k \neq 0$, is mapped via the local langlands correspondence into a discrete series $D_{|\ell_k|}$. At the level of K -theory, a generator $[D_{|\ell_k|}]$ is mapped into $[\chi_{|\ell_k|}]$. The result follows from (16).

Case 2: $m > 0$ odd

Then, if $n \equiv j \pmod{2}$, $K^j(\mathbb{R}^{n+m}) = 0$ and $K_j(\mathcal{AI})$ is zero.

Case 3: $m > 0$ is even

In this case $K^j(\mathbb{R}^n) = K^j(\mathbb{R}^{n+m})$. However, $X_{\mathbb{R}}(M) \simeq \mathbb{R}^{n+m}$ corresponds precisely to the partition of $2n$ into 1's and 0's given by

$$2n = 2(n - m) + 2m$$

Hence, the number of 1's in the partition is $2m \geq 4$. It follows that (t_1, \dots, t_n) is mapped into a cone and, as a consequence, $K_j(\mathcal{AI})$ is zero.

This concludes the proof. \square

7 Connections with the Baum-Connes correspondence

The standard maximal compact subgroup of $GL(1, \mathbb{C})$ is the circle group $U(1)$, and the maximal compact subgroup of $GL(1, \mathbb{R})$ is $\mathbb{Z}/2\mathbb{Z}$. Base change for K^1 creates a map

$$\mathcal{R}(U(1)) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{R}(U(1))$ is the representation ring of the circle group $U(1)$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $U(1)$ to $1 \oplus \varepsilon$, where ε is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $U(1)$ to zero.

This map has an interpretation in terms of K -cycles. The real line \mathbb{R} is a universal example for the action of \mathbb{R}^\times and \mathbb{C}^\times . The K -cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to \mathbb{C}^\times and \mathbb{R}^\times , and therefore determines a class $\mathcal{J}_{\mathbb{C}} \in K_1^{\mathbb{C}^\times}(E\mathbb{C}^\times)$ and a class $\mathcal{J}_{\mathbb{R}} \in K_1^{\mathbb{R}^\times}(E\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\mathcal{J}_{\mathbb{C}} \mapsto (\mathcal{J}_{\mathbb{R}}, \mathcal{J}_{\mathbb{R}})$$

It would be interesting to interpret the automorphic induction map at the level of representation rings:

$$\mathcal{AI}^* : \mathcal{R}(O(2n)) \longrightarrow \mathcal{R}(U(n)).$$

References

- [1] J. Arthur, L. Clozel, Simple algebras, base change, and the advanced theory of the trace formula, *Annals of Math. Studies* 120, Princeton University Press, Princeton, 1989.
- [2] P. Baum, A. Connes, N. Higson, Classifying space for proper actions and K -theory for group C^* -algebras, *Contemporary Math.*, 167 (1994) 241–291.
- [3] J. Bernstein, S. Gelbart et al, An introduction to the Langlands program, Birkhauser 2003.
- [4] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York 1982.
- [5] J. Chabert, S. Echterhoff, R. Nest, The Connes-Kasparov conjecture for almost connected groups and for linear p -adic groups, *Publications Math. I.H.E.S.* 97 (2003) 239–278.
- [6] Harish-Chandra, *Collected papers*, Vol. 4, Springer, Berlin (1984).

- [7] G. Henniart, Induction automorphe pour $GL(n, \mathbb{C})$, J. Functional Analysis 258, issue 9 (2010) 3082–3096.
- [8] A. W. Knap, Local Langlands correspondence: the archimedean case, Proc. Symp. Pure. Math. 55, 1994, 393–410.
- [9] V. Lafforgue, K -théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes, Invent. Math. 149 (2002) 1–95.
- [10] C. Mœglin, Representations of $GL(n)$ over the Real Field, Proc. Symp. Pure Math. 61 (1997), 157–166.
- [11] S. Mendes, R.J. Plymen, Base change and K -theory for $GL(n)$, J. Non-commut. Geom., 1 (2007), 311–331.
- [12] R. J. Plymen, The reduced C^* -algebra of the p -adic group $GL(n)$, J. Functional Analysis 72 (1987) 1–12.
- [13] M. G. Penington, R. J. Plymen, The Dirac operator and the principal series for complex semisimple Lie groups, J. Functional Analysis, 53 (1983) 269–286.

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